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The Levi-Civita equation in function classes

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Dedicated to the 95th birthday of János Aczél.

Abstract. Let G be an Abelian topological semigroup with unit. By a classical result (called Theorem A), if V is a finite dimensional translation invariant linear space of complex valued continuous functions defined on G, then every element of V is an exponential polynomial. More precisely, every element of V is of the form $\sum_{i=1}^{n} p_i \cdot m_i$, where $m_1, \ldots, m_n$ are exponentials belonging to V, and $p_1, \ldots, p_n$ are polynomials of continuous additive functions. We generalize this statement by replacing the set of continuous functions by any algebra $\mathcal{A}$ of complex valued functions such that whenever an exponential m belongs to $\mathcal{A}$, then $m^{-1} \in \mathcal{A}$. As special cases we find that Theorem A remains valid even if the topology on G is not compatible with the operation on G, or if the set of continuous functions is replaced by the set of measurable functions with respect to an arbitrary σ-algebra. We give two proofs of the result. The first is based on Theorem A. The second proof is independent, and seems to be more elementary than the existing proofs of Theorem A.

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1. Introduction and main results

Let G be an Abelian topological semigroup with unit, and let $C(G)$ denote the set of complex valued continuous functions defined on G. By a *polynomial* we mean a function of the form $P(\alpha_1, \ldots, \alpha_k)$, where $P \in \mathbb{C}[x_1, \ldots, x_k]$ and $\alpha_1, \ldots, \alpha_k$ are continuous additive functions. An *exponential* is a function $m \in C(G)$ such that $m \neq 0$ and $m(x+y) = m(x) \cdot m(y)$ for every $x, y \in G$. Functions of the form $\sum_{i=1}^{n} p_i \cdot m_i$, where p_i is a polynomial and m_i is an exponential for every $i = 1, \ldots, n$, are called *exponential polynomials*. In these representations we may assume that $p_i \neq 0$ for every i, and the exponentials $m_1, \ldots, m_n$ are distinct. One can show that such a representation is unique (see [15, Lemma 4.3, p. 41] or [6, Lemma 6]). Our starting point is the following well-known result.

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Theorem A. *Let G be an Abelian topological semigroup with unit. If V is a finite dimensional translation invariant linear subspace of $C(G)$, then every element of V is an exponential polynomial. More precisely, every element of V is of the form $\sum_{i=1}^{k} p_i \cdot m_i$, where $p_1, \ldots, p_k$ are polynomials, and $m_1, \ldots, m_k$ are exponentials belonging to V.*

An equivalent form of the theorem above states that if a continuous function $f \in C(G)$ satisfies the equation $f(x+y) = \sum_{i=1}^{n} f_i(x) \cdot g_i(y)$ for every $x, y \in G$ with suitable functions $f_i, g_i : G \to \mathbb{C}$, then f is an exponential polynomial. For continuously differentiable functions on $\mathbb{R}$ this was proved by Levi-Civita in 1913 [10]. For continuous functions defined on $\mathbb{R}$ a much more general statement was proved by L. Schwartz in 1947 (see [12]). A simple proof in the case of $G = \mathbb{R}$ was given by Anselone and Korevaar in 1964; see [3] and also [9]. The generalization for Abelian (topological) semigroups was treated by Stone in 1960 and McKiernan in 1977; see [13] and [11]. The case of Abelian topological groups was also discussed by Laird and by Székelyhidi, see [8], [15, Theorem 10.1], [16]. (As for the history of Theorem A, see also [2] and the references given there.) The result was generalized for measurable (instead of continuous) functions defined on locally compact Abelian groups, see [5,14] and [15, Theorem 10.2].

Let G be an Abelian semigroup with unit, and let $\mathbb{C}^G$ denote the set of complex valued functions defined on G. The discrete topology makes G a topological semigroup with $C(G) = \mathbb{C}^G$. A function p will be called a *discrete polynomial*, if it is a polynomial w.r.t. the discrete topology; that is, if $p = P(\alpha_1, \ldots, \alpha_k)$, where $P \in \mathbb{C}[x_1, \ldots, x_k]$ and $\alpha_1, \ldots, \alpha_k : G \to \mathbb{C}$ are additive functions. Functions of the form $\sum_{i=1}^{k} p_i \cdot m_i$, where $p_1, \ldots, p_k$ are discrete polynomials and $m_1, \ldots, m_k$ are exponentials are called *discrete exponential polynomials.*

From Theorem A it follows that if V is a finite dimensional translation invariant linear subspace of $\mathbb{C}^G$, then every element of V is a discrete exponential polynomial. Moreover, the exponential functions appearing in the representations of the elements of V belong to V.

Our aim is to generalize Theorem A by replacing the set of continuous (or measurable) functions by a subalgebra of $\mathbb{C}^G$. Suppose a subalgebra $\mathcal{A}$ of $\mathbb{C}^G$ is given. We say that p is an *admissible polynomial* with respect to $\mathcal{A}$, if $p = P(\alpha_1, \ldots, \alpha_k)$, where $P \in \mathbb{C}[x_1, \ldots, x_k]$ and $\alpha_1, \ldots, \alpha_k$ are additive functions belonging to $\mathcal{A}$.

Looking for a generalization of Theorem A for function classes, the following question arises. Suppose V is a finite dimensional translation invariant linear subspace of the given algebra $\mathcal{A}$. We know that every element of V is of the form $\sum_{i=1}^{k} p_i \cdot m_i$, where $p_1, \ldots, p_k$ are discrete polynomials, and $m_1, \ldots, m_k \in V$ are exponentials. When can we expect that in these representations $p_1, \ldots, p_n$ are admissible w.r.t. $\mathcal{A}$?

If the algebra $\mathcal{A}$ contains the constant functions, then every polynomial admissible w.r.t. $\mathcal{A}$ is an element of $\mathcal{A}$. Therefore, a necessary condition for such a generalization to hold is that if p is a discrete polynomial and m is an exponential, and the translates of $p \cdot m$ belong to $\mathcal{A}$, then $p \in \mathcal{A}$. This is not true for every algebra. For example, let G be the additive group of $\mathbb{R}$, and let

$$\mathcal{A} = \left\{ c + \sum_{i=1}^{n} p_i \cdot e^{ix} : n \geq 0,\ c \in \mathbb{R},\ p_1, \ldots, p_n \in \mathbb{C}[x] \right\}.$$

Then $\mathcal{A}$ is an algebra over $\mathbb{C}$ containing the constant functions, the translates of $x \cdot e^x$ belong to $\mathcal{A}$, but $x \notin \mathcal{A}$.

We show that the following simple condition is sufficient:

$$\text{if } m \in \mathcal{A} \text{ is an exponential, then } m^{-1} \in \mathcal{A}. \tag{1}$$

Clearly, (1) is satisfied if $\mathcal{A}$ contains the constant functions and $\mathcal{A}$ is closed under division; that is, if $f, g \in \mathcal{A}$ and $g \neq 0$ implies $f/g \in \mathcal{A}$.

In particular, if $\mathcal{T}$ is a topology on G (not necessarily compatible with the group operations), then the set of continuous functions with respect to $\mathcal{T}$ is an algebra satisfying (1), and so is the set of Borel measurable functions. Or, if Σ is an arbitrary σ-algebra on G, then the set of functions measurable with respect to Σ is also an algebra satisfying (1).

If G is an Abelian group and m is an exponential on G, then we have $m^{-1}(x) = m(-x)$ for every $x \in G$. Therefore, condition (1) is also satisfied if G is an Abelian group and the algebra $\mathcal{A}$ is *symmetric*; that is, if $f \in \mathcal{A}$ implies $\hat{f} \in \mathcal{A}$, where $\hat{f}(x) = f(-x)$ for every $x \in G$.

Our main result is the following.

Theorem 1. *Let G be an Abelian semigroup with unit, and let $\mathcal{A} \subset \mathbb{C}^G$ be an algebra over $\mathbb{C}$ satisfying condition (1). If V is a finite dimensional translation invariant linear subspace of $\mathcal{A}$, then every element of V is of the form $\sum_{i=1}^{k} p_i \cdot m_i$, where $p_1, \ldots, p_k$ are admissible polynomials w.r.t. $\mathcal{A}$, and $m_1, \ldots, m_k$ are exponentials belonging to V.*

Theorem 1 is an easy consequence of Theorem A and the following result.

Theorem 2. *Let $\mathcal{A}$ be a subalgebra of $\mathbb{C}^G$. A function f is an admissible polynomial w.r.t. $\mathcal{A}$ if and only if f is a discrete polynomial, and all translates of f belong to $\mathcal{A}$.*

Note the following special case. *If G is a topological Abelian semigroup with unit, then $f\colon G \to \mathbb{C}$ is a polynomial if and only if f is a continuous discrete polynomial.*

Taking Theorem 2 for granted, we can prove Theorem 1 as follows. Let V be a finite dimensional translation invariant linear subspace of an algebra $\mathcal{A}$ satisfying condition (1). By Theorem A, every element $f \in V$ is of the form $f = \sum_{i=1}^{n} p_i \cdot m_i$, where $p_1, \ldots, p_n$ are nonzero discrete polynomials, and

$m_1, \ldots, m_n$ are distinct exponentials belonging to V. Then we have $p_i \cdot m_i \in V$ and $m_i \in V$ for every $i = 1, \ldots, n$ (see [6, Lemma 6]). Fix an index i. Since $m_i \in V \subset \mathcal{A}$, it follows from condition (1) that $p_i \in \mathcal{A}$. By the translation invariance of V we have

$$T_h f = \sum_{i=1}^{n} m_i(h) \cdot T_h p_i \cdot m_i \in V$$

for every $h \in G$. This implies, by the argument above, that $T_h p_i \in \mathcal{A}$ for every $h \in G$. Thus the translates of p_i belong to $\mathcal{A}$. Therefore, by Theorem 2, we obtain that p_i is admissible w.r.t. $\mathcal{A}$. $\square$

In the argument above, we only used the 'if' part of Theorem 2, which is an immediate consequence of the following result.

Theorem 3. *Let G be an Abelian semigroup, and let $\mathcal{P}_G$ denote the algebra of discrete polynomials on G. A subalgebra $\mathcal{A} \subset \mathcal{P}_G$ is translation invariant if and only if $\mathcal{A} = \{0\}$ or $\mathcal{A}$ is generated by a set of additive functions and the constant functions.*

Now suppose that the translates of a discrete polynomial f belong to an algebra $\mathcal{A}$. We may assume $f \neq 0$. Let $\mathcal{A}_f$ denote the algebra generated by the translates of f. Then $\mathcal{A}_f \subset \mathcal{A}$ and $\mathcal{A}_f$ is translation invariant. By Theorem 3, $\mathcal{A}_f$ is generated by a set of additive functions and the constant functions. Since $f \in \mathcal{A}_f$, this implies that $f = P(b_1, \ldots, b_k)$, where $P \in \mathbb{C}[x_1, \ldots, x_k]$, and $b_1, \ldots, b_k$ are additive functions belonging to $\mathcal{A}_f$, hence also to $\mathcal{A}$. This means that f is admissible w.r.t. $\mathcal{A}$, proving the 'if' part of Theorem 2.

The proof of Theorem 1 will be completed in the next section, where we prove Theorems 2 and 3.

In Sect. 3 we also give a direct proof of Theorem 1 using none of Theorem A or Theorems 2 and 3. The reason why we present such an independent proof is that it seems to be more elementary than the existing proofs of Theorem A, which is a special case of Theorem 1. We also show that Theorems 2, 3, and some other results we present in the next section are easy consequences of Theorem 1.

2. Proof of Theorems 2 and 3

If G is an Abelian semigroup, $h \in G$ and $f\colon G \to \mathbb{C}$, then we denote by $T_h f$ the function $x \mapsto f(x+h)$ $(x \in G)$. We put $\Delta_h f = T_h f - f$ for every $h \in G$. If $f\colon G \to \mathbb{C}$, then we denote by V_f and $\mathcal{A}_f$, respectively, the linear space and the algebra generated by the set $\{T_h f\colon h \in G\}$.

The set of complex valued linear functions defined on $\mathbb{C}^n$ will be denoted by Λ_n.

Theorem 4. (i) *Every homogeneous polynomial* $f \in \mathbb{C}[x_1, \ldots, x_n]$ *can be represented in the form* $p(\ell_1, \ldots, \ell_k)$*, where* $k \le n$*,* $p \in \mathbb{C}[x_1, \ldots, x_k]$ *and* $\ell_1, \ldots, \ell_k \in V_f \cap \Lambda_n$*.*

(ii) *Every polynomial* $f \in \mathbb{C}[x_1, \ldots, x_n]$ *can be represented in the form* $p(\ell_1, \ldots, \ell_k)$*, where* $k \le n$*,* $p \in \mathbb{C}[x_1, \ldots, x_k]$ *and* $\ell_1, \ldots, \ell_k \in \mathcal{A}_f \cap \Lambda_n$*.*

Proof. We prove (ii) by induction on $\deg f$ and, along the proof, we also prove (i). Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be arbitrary. Statement (ii) is true if f is constant. Let $\deg f = d > 0$, and suppose that (ii) is true for all polynomials of degree $< d$. It is easy to see that the constant functions belong to V_f. Indeed, $\Delta_{h_1} \cdots \Delta_{h_d} f$ is a nonzero constant function for suitable $h_1, \ldots, h_d \in \mathbb{C}^n$, and belongs to V_f. Since V_f is a linear space, it follows that V_f contains every constant function.

Obviously, Λ_n is an n dimensional linear space over the complex field. The set $V_f \cap \Lambda_n$ is a linear subspace of Λ_n, and thus its dimension, k, is at most n. Let $b_i = \beta_{i,1}x_1 + \cdots + \beta_{i,n}x_n$ $(i = 1, \ldots, k)$ be a basis of $V_f \cap \Lambda_n$. Since $b_1, \ldots, b_k$ are linearly independent, the rank of the matrix $A = (\beta_{i,j})$ $(i = 1, \ldots, k,\ j = 1, \ldots, n)$ equals k, and thus A contains a $k \times k$ submatrix with nonzero determinant. We may assume that the determinant $|\beta_{i,j}|_{i,j=1,\ldots,k}$ is nonzero, since otherwise we may apply a suitable permutation of the variables $x_1, \ldots, x_n$. Then there is an invertible system of linear substitutions

$$x_i = \gamma_{i,1}y_1 + \cdots + \gamma_{i,n}y_n \qquad (i = 1, \ldots, n) \tag{2}$$

that transforms b_i to y_i for every $i = 1, \ldots, k$. (Put $x_i = y_i$ for every $k < i \le n$, and solve the system of equations $b_i = y_i$ $(i = 1, \ldots, k)$ for $x_1, \ldots, x_k$.)

Let $p \in \mathbb{C}[y_1, \ldots, y_n]$ be the polynomial obtained from f by substitution (2). It is easy to check that substitution (2) induces a bijection from V_f onto V_p. Since $b_i \in V_f$ for every $i = 1, \ldots, k$, it follows that $y_1, \ldots, y_k \in V_p$. Considering that $b_1, \ldots, b_k$ is a basis of $V_f \cap \Lambda_n$, we find that a linear function belongs to V_p if and only if it involves the variables $y_1, \ldots, y_k$ only.

Let f_d and p_d denote, respectively, the sum of all terms of f and p with degree d. It is clear that p_d is obtained from f_d by substitution (2).

Let $p = \sum_i c_i \cdot y_1^{i_1} \cdots y_n^{i_n}$, where $i = (i_1, \ldots, i_n)$ runs through the n-tuples $(i_1, \ldots, i_n)$ with $i_1 + \cdots + i_n \le d$. We claim that $p_d \in \mathbb{C}[y_1, \ldots, y_k]$; that is, none of the variables $y_{k+1}, \ldots, y_n$ occur in a term of p_d with nonzero coefficient. Indeed, suppose that $c_i \cdot y_1^{i_1} \cdots y_n^{i_n}$ is a term in p_d such that $c_i \ne 0$, $i_1 + \cdots + i_n = d$, and there is an index $j > k$ with $i_j > 0$. We may assume that $j = n$. Let $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ be the standard basis of $\mathbb{C}^n$, and let D denote the operator $\Delta_{e_1}^{i_1} \cdots \Delta_{e_{n-1}}^{i_{n-1}} \Delta_{e_n}^{i_n - 1}$. Clearly, $Dp \in V_p$. Let $c_j \cdot y_1^{j_1} \cdots y_n^{j_n}$ be an arbitrary term in p. Then its image under D is constant unless $j_1 \ge i_1, \ldots, j_{n-1} \ge i_{n-1}$, and $j_n \ge i_n - 1$, and at least one of the inequalities is strict. The images of these terms are distinct, and are of the form $\gamma_i y_i + \delta_i$, where $\gamma_i, \delta_i \in \mathbb{C}$. Now one of these terms equals $\gamma_n y_n + \delta_n$ with a nonzero γ_n, and thus $Dp = \gamma_1 y_1 + \cdots + \gamma_n y_n +$ constant, where $\gamma_n \ne 0$. Since the constant functions belong to V_p, it follows that $\gamma_1 y_1+\dots+\gamma_n y_n \in V_p \cap \Lambda_n$. This, however, contradicts the fact that all linear functions in V_p involve the variables $y_1,\dots,y_k$ only. Therefore, we have $p_d \in \mathbb{C}[y_1,\dots,y_k]$.

Since p_d is obtained from f_d by substitution (2) and the substitution is invertible and transforms b_i to y_i $(1 \le i \le k)$, it follows that

$$f_d = p_d(b_1,\dots,b_k). \tag{3}$$

Now $b_1,\dots,b_k$ are linear functions belonging to V_f. Therefore, if f is homogeneous, then $f = f_p$, and (i) holds true.

In the general case let $\mathcal{L}_f$ denote the algebra generated by $\mathcal{A}_f \cap \Lambda_n$ and the constant functions. We have to prove $f \in \mathcal{L}_f$.

By (3), we have $f_d \in \mathcal{L}_f \subset \mathcal{A}_f$. Putting $g = f - f_p$ we find $g \in \mathcal{A}_f$, and thus we have $\mathcal{A}_g \subset \mathcal{A}_f$. Since $\deg g < d$, it follows from the induction hypothesis that g belongs to the algebra generated by $\mathcal{A}_g \cap \Lambda_n$ and the constant functions. Since $\mathcal{A}_g \cap \Lambda_n \subset \mathcal{A}_f \cap \Lambda_n$, we have $g \in \mathcal{L}_f$. Therefore, by $f_p \in \mathcal{L}_f$ we obtain $f = f_d + g \in \mathcal{L}_f$. $\square$

Remark 5. In statement (ii) of Theorem 4 we cannot replace $\mathcal{A}_f \cap \Lambda_n$ by $V_f \cap \Lambda_n$, as the following example shows. Let $n = 2$ and $f(x,y) = x^2 + y \in \mathbb{C}[x,y]$. It is easy to check that

$$V_f = \{a \cdot x^2 + a \cdot y + bx + c \colon a,b,c \in \mathbb{C}\}.$$

Thus the linear functions belonging to V_f are the functions bx $(b \in \mathbb{C})$. Clearly, f is not in the algebra generated by these functions and the constants.

Lemma 6. *Suppose $c_1,\dots,c_n \in \mathbb{C}^n$ are linearly independent over $\mathbb{C}$. If a linear subspace V of $\mathbb{C}[x_1,\dots,x_n]$ is invariant under the translations T_{c_i} $(i = 1,\dots,n)$, then V is invariant under all translations.*

Proof. Let H denote the additive group generated by $c_1,\dots,c_n$. Then V is invariant under translations by elements of H. Let $e_1,\dots,e_n$ be the standard basis of $\mathbb{C}^n$, and let $A\colon \mathbb{C}^n \to \mathbb{C}^n$ be the linear transformation mapping e_i to c_i $(i = 1,\dots,n)$. Then A maps $\mathbb{Z}^n$ onto H.

The set $V' = \{p \circ A \colon p \in V\}$ is also a linear subspace of $\mathbb{C}[x_1,\dots,x_n]$. If $p \in V$ and $a \in \mathbb{Z}^n$, then $T_a(p \circ A)(x) = p(A(x) + h) = q(A(x))$, where $h \in H$ and $q = T_h p$. By assumption, $q \in V$, and thus $T_a(p \circ A) \in V'$. Therefore, V' is invariant under translations by the elements of $\mathbb{Z}^n$. By [7, Lemma 7], this implies that V' is invariant under all translations. Thus the same is true for $V = \{q \circ A^{-1} \colon q \in V'\}$. $\square$

Proof of Theorem 3. Let G be an arbitrary Abelian semigroup, and suppose that the algebra $\mathcal{A}$ is generated by a set $\mathcal{L}$ of additive functions and the constant functions. If $a \in \mathcal{L}$, then $a(x+h) = a(x) + a(h)$ for every $x \in G$, and thus $T_h a = a + a(h) \in \mathcal{A}$. From this observation it is clear that $\mathcal{A}$ is translation invariant.

Now let $\mathcal{A} \subset \mathcal{P}_G$ be a nonzero translation invariant subalgebra. Then $\mathcal{A}$ contains the constant functions. Indeed, for every $f \in \mathcal{A}$, $f \neq 0$, already V_f (a subset of $\mathcal{A}$) contains all constant functions.

Let $f \in \mathcal{A}$ be arbitrary. Then $f = p(a_1, \dots, a_n)$, where $p \in \mathbb{C}[x_1, \dots, x_n]$, and $a_1, \dots, a_n \colon G \to \mathbb{C}$ are additive. We may assume that $a_1, \dots, a_n$ are linearly independent over $\mathbb{C}$. Then there are points $h_1, \dots, h_n \in G$ such that the determinant $|a_i(h_j)|$ $(i, j = 1, \dots, n)$ is nonzero (see [1, Lemma 1, p. 229]). Let $c_j = (a_1(h_j), \dots, a_k(h_j))$ $(j = 1, \dots, n)$. Then the vectors $c_1, \dots, c_n \in \mathbb{C}^n$ are linearly independent over $\mathbb{C}$.

Let V denote the set of polynomials $q \in \mathbb{C}[x_1, \dots, x_n]$ such that $q(a_1, \dots, a_n) \in \mathcal{A}$. Then V is a subalgebra of $\mathbb{C}[x_1, \dots, x_n]$. We prove that V is translation invariant. If $q \in V$, then $q(a_1, \dots, a_n) \in \mathcal{A}$ and thus, as $\mathcal{A}$ is translation invariant, we have $T_{h_j} q(a_1, \dots, a_n) \in \mathcal{A}$ for every $j = 1, \dots, n$. It is clear that $T_{h_j} q(a_1, \dots, a_n) = (T_{c_j} p)(a_1, \dots, a_n)$, and thus $T_{c_j} q \in V$ for every $q \in V$ and $j = 1, \dots, n$. By Lemma 6, this implies that V is translation invariant.

Since $p(a_1, \dots, a_n) = f \in \mathcal{A}$, we have $p \in V$. Then, by the translation invariance of V, we have $\mathcal{A}_p \subset V$. By (ii) of Theorem 4, $p = P(\ell_1, \dots, \ell_k)$, where $k \leq n$, $P \in \mathbb{C}[x_1, \dots, x_k]$, and

$$\ell_1, \dots, \ell_k \in \mathcal{A}_p \cap \Lambda_n \subset V \cap \Lambda_n.$$

Now $\ell_i \in V$ implies that $b_i = \ell_i(a_1, \dots, a_n) \in \mathcal{A}$ for every $i = 1, \dots, k$. Thus we have

$$f = p(a_1, \dots, a_n) = P(\ell_1(a_1, \dots, a_n), \dots, \ell_k(a_1, \dots, a_n)) = P(b_1, \dots, b_k),$$

where $b_1, \dots, b_k$ are additive functions belonging to $\mathcal{A}$. Since $f \in \mathcal{A}$ was arbitrary, this proves that $\mathcal{A}$ equals the algebra generated by the set of additive functions belonging to $\mathcal{A}$ and the constant functions. $\square$

Proof of Theorem 2. The 'if' part was proved in Sect. 1. To prove the 'only if' part, let f be an admissible polynomial w.r.t. $\mathcal{A}$. Let $f = P(\alpha_1, \dots, \alpha_k)$, where $P \in \mathbb{C}[x_1, \dots, x_k]$ and $\alpha_1, \dots, \alpha_k$ are additive functions belonging to $\mathcal{A}$. Then f belongs to the algebra $\mathcal{B}$ generated by $\alpha_1, \dots, \alpha_k$ and the constant functions. By Theorem 3, $\mathcal{B}$ is translation invariant. Therefore, every translate of f belongs to $\mathcal{B} \subset \mathcal{A}$. $\square$

3. A direct proof of Theorem 1

Let G be an Abelian semigroup with unit. A function $f \colon G \to \mathbb{C}$ is a *generalized polynomial*, if there is an $n \geq 0$ such that $\Delta_{h_1} \cdots \Delta_{h_{n+1}} f = 0$ for every $h_1, \dots, h_{n+1} \in G$. The smallest n with this property is the *degree* of f denoted by $\deg f$. The degree of the identically zero function is -1 by definition. One can easily prove that every discrete polynomial is a generalized polynomial.

A function $A\colon G^i \to \mathbb{C}$ is called i-additive, if it is additive in each of its variables, the other variables being fixed. A function $g\colon G \to \mathbb{C}$ is a *monomial of degree* i, if there is an i-additive function A such that $g(x) = A(x,\dots,x)$ for every $x \in G$. It is well-known that every generalized polynomial can be represented in the form $\sum_{i=0}^{d} f_i$, where f_i is a monomial of degree i for every $i = 1,\dots,d$, and f_0 is constant. (See [4, Theorem 3].)

We fix an algebra $\mathcal{A} \subset \mathbb{C}^G$ satisfying condition (1). By an admissible polynomial we mean an admissible polynomial w.r.t. $\mathcal{A}$.

Lemma 7. *Let f be an admissible polynomial, and let $f = \sum_{i=0}^{n} f_i$, where f_i is a monomial of degree i for $1 \le i \le n$, and f_0 is a constant. Then f_i is an admissible polynomial for every $i = 0,\dots,n$.*

Proof. It is easy to see that $f(kx) = \sum_{i=0}^{n} k^i \cdot f_i(x)$ for every $x \in G$ and for every positive integer k. These equations for $k = 1,\dots,n+1$ constitute a linear system of equations with unknowns $f_i(x)$ $(i = 0,\dots,n)$. Since the determinant of this system is nonzero (being a Vandermonde determinant), it follows that each $f_i(x)$ is a linear combination with rational coefficients of $f(x),\dots,f((n+1)x)$. It is clear that each of the functions $f(x),\dots,f((n+1)x)$ is an admissible polynomial, and thus the same is true of $f_i(x)$ $(i = 0,\dots,n)$. $\square$

Lemma 8. *Suppose that $\mathcal{A}$ contains the constant functions. If f is a generalized polynomial and $x \mapsto f(2x) - 2f(x)$ is an admissible polynomial, then f is the sum of an admissible polynomial and an additive function. If, in addition, $f \in \mathcal{A}$, then f is an admissible polynomial.*

Proof. Let $f = \sum_{i=0}^{n} f_i$, where f_i is a monomial of degree i for every $1 \le i \le n$, and f_0 is a constant. Then

$$f(2x) - 2f(x) = -a_0 + \sum_{i=2}^{n} (2^i - 2) f_i(x)$$

for every $x \in G$. Since $f(2x) - 2f(x)$ is an admissible polynomial by assumption, it follows from Lemma 7 that f_i is an admissible polynomial for every $i \ne 1$. As f_1 is additive, we obtain the first statement of the lemma. If $f \in \mathcal{A}$, then $f_1 = f - \sum_{i \ne 1} f_i \in \mathcal{A}$. Since f_1 is additive, it follows that f_1 is an admissible polynomial, and then so is f. $\square$

Lemma 9. *Let V be a translation invariant and finite dimensional linear subspace of $C(G)$. If $V \ne 0$, then V contains an exponential.*

Proof. Although the statement is well-known, we provide the proof for the sake of completeness. We prove by induction on the dimension of V. Let $\dim V = n$, and suppose that either $n = 1$, or $n > 1$ and the statement is true for smaller positive dimensions.

Let $h \in G$ be fixed. Then the translation operator T_h is a linear transformation mapping V into itself. Since $\dim V < \infty$, it follows that T_h has an eigenvalue $\lambda(h)$. Then $L_h = T_h - \lambda(h) \cdot T_0$ is not invertible and, consequently, the image space $\operatorname{Im} L_h$ is a proper subspace of V. It is clear that $\operatorname{Im} L_h$ is a translation invariant linear subspace of V. If $\operatorname{Im} L_h \neq 0$ then, by the induction hypothesis, $\operatorname{Im} L_h$ contains an exponential, and we are done. Therefore, we may assume that $\operatorname{Im} L_h = 0$ for every h; that is, $T_h f = \lambda(h) f$ for every $h \in G$ and $f \in V$.

Let $f \in V$, $f \neq 0$ be fixed. Then we have $f(x+h) = \lambda(h) f(x)$ for every $x \in G$. Thus $f(h) = c \cdot \lambda(h)$ for every h, where $c = f(0)$.[1] Then $c \neq 0$, since otherwise f would be identically zero. Therefore, we find that $\lambda \in V$, $\lambda \neq 0$ and $\lambda(x+h) = \lambda(h)\lambda(x)$ for every $x, h \in G$; that is, λ is an exponential. $\square$

Proof of Theorem 1. We prove by induction on the dimension of V. The statement is true if $V = 0$. Let $\dim V > 0$, and suppose that the statement is true for smaller dimensions.

Let W denote the set of those elements of V that can be written in the form $\sum_{i=1}^{k} p_i \cdot m_i$, where $p_1, \ldots, p_k$ are admissible polynomials and $m_1, \ldots, m_k$ are distinct exponentials belonging to V. It is easy to see that W is a translation invariant linear subspace of V. We have to prove that $W = V$. Suppose this is not true, and fix an element $f_0 \in V \setminus W$.

Let m be an arbitrary exponential contained in V, and let $h \in G$ be fixed. Then $L_h = T_h - m(h) \cdot T_0$ is a linear transformation mapping V into itself. Since $\dim V < \infty$ and the kernel of L_h is nonzero as $L_h(m) = 0$, it follows that the image space $\operatorname{Im} L_h$ of L_h is a proper subspace of V. Clearly, $\operatorname{Im} L_h$ is translation invariant and thus, by the induction hypothesis, every element of $\operatorname{Im} L_h$ is of the form $\sum_{i=1}^{k} p_i \cdot m_i$, where $p_1, \ldots, p_k$ are admissible polynomials and $m_1, \ldots, m_k$ are exponentials belonging to $\operatorname{Im} L_h$. Therefore, we have $\operatorname{Im} L_h \subset W$ for every h.

We show that $\langle f_0, W \rangle = \{c \cdot f_0 + p \colon c \in \mathbb{C},\ p \in W\}$ is a translation invariant linear subspace of V. It is clear that $\langle f_0, W \rangle$ is a linear subspace of V. Its invariance under translations follows from the facts that W is translation invariant, $T_h f_0 = m(h) \cdot f_0 + L_h(f_0)$, and $L_h(f_0) \in \operatorname{Im} L_h \subset W$.

If $\langle f_0, W \rangle$ is a proper subspace of V then, by the induction hypothesis, $\langle f_0, W \rangle \subset W$, and thus $f_0 \in W$, a contradiction.

Therefore, we have $\langle f_0, W \rangle = V$; that is, every element of V is of the form $c \cdot f_0 + p$ ($c \in \mathbb{C},\ p \in W$). Let $b_1, \ldots, b_k$ be a basis of W. Since $L_y f_0 \in W$, we

[1] It is this point where we need the condition that G has a unit. Note that without this condition neither the lemma nor Theorem 1 remains valid. Let G be the set of positive integers with the usual addition, and put $f_0(1) = 1$, $f_0(n) = 0$ ($n > 1$). Then $V = \{c f_0 \colon c \in \mathbb{C}\}$ is a one dimensional subspace of $\mathbb{C}^G$, and V is translation invariant, as $T_h f = 0$ for every $f \in V$ and $h \in G$. However, every element of V vanishes at 2, and thus V does not contain an exponential.

have

$$f_0(x+y) - m(y)f_0(x) = L_y f_0(x) = \sum_{i=1}^{k} b_i(x) \cdot q_i(y) \tag{4}$$

for every $x, y \in G$, with suitable functions $q_i \colon G \to \mathbb{C}$.

We show that $q_1, \ldots, q_k \in V$. Since $b_1, \ldots, b_k$ are linearly independent, there are points $x_1, \ldots, x_k \in G$ such that the determinant $|b_i(x_j)|$ $(i, j = 1, \ldots, k)$ is nonzero (see [1, Lemma 1, p. 229]). Applying (4) with $x = x_j$ for every $j = 1, \ldots, k$ we obtain a system of linear equations with unknowns $q_i(y)$. Since the determinant of the system is nonzero, it follows that each of $q_1(y), \ldots, q_k(y)$ is a linear combination of $f_0(x_j+y) - m(y)f_0(x_j)$ $(j = 1, \ldots, k)$. The functions $y \mapsto f_0(x_j + y) - m(y)f_0(x_j)$ belong to V, and then the same is true for $q_1, \ldots, q_k$.

Let $q_i = c_i \cdot f_0 + p_i$, where $c_i \in \mathbb{C}$ and $p_i \in W$ for every $i = 1, \ldots, k$. Then, by (4), we have

$$\begin{aligned} f_0(x+y) - m(y)f_0(x) &= \sum_{i=1}^{k} b_i(x) \cdot (c_i \cdot f_0(y) + p_i(y)) \\ &= f_0(y) \cdot \sum_{i=1}^{k} c_i \cdot b_i(x) + \sum_{i=1}^{k} b_i(x) \cdot p_i(y) \end{aligned}$$

and

$$f_0(x+y) - m(y)f_0(x) - m(x)f_0(y) = B(x) \cdot f_0(y) + \sum_{i=1}^{k} b_i(x) \cdot p_i(y) \tag{5}$$

for every $x, y \in G$, where $B(x) = -m(x) + \sum_{i=1}^{k} b_i(x)$. If x is fixed, then the left hand side of (5), as a function of y, equals $L_x f_0 - f_0(x) \cdot m \in W$. Thus the right hand side of (5) also belongs to W. Since $p_i \in W$ for every i, it follows that $B(x) \cdot f_0 \in W$. Since $f_0 \notin W$, it follows that $B(x) = 0$ for every x, and thus

$$f_0(x+y) - m(y)f_0(x) - m(x)f_0(y) = \sum_{i=1}^{k} b_i(x) \cdot p_i(y) \tag{6}$$

for every $x, y \in G$.

Suppose that V contains two different exponentials m' and m. Then the argument above, with m' in place of m, gives

$$f_0(x+y) - m'(y)f_0(x) - m'(x)f_0(y) = \sum_{i=1}^{k} b_i(x) \cdot p_i'(y)$$

for every $x, y \in G$, where $p_i'(y) \in W$ for every $i = 1, \dots, k$. Subtracting (6) we get

$$(m(y) - m'(y)) \cdot f_0(x) + f_0(y) \cdot (m(x) - m'(x)) = \sum_{i=1}^{k} b_i(x) \cdot (p_i'(y) - p_i(y)).$$

If y is fixed, then the right hand side, as a function of x, belongs to W, and so does $f_0(y) \cdot (m(x) - m'(x))$. Consequently, if we fix y such that $m(y) \neq m'(y)$, then we find $f_0 \in W$, which is impossible.

Therefore, V contains at most one exponential. Now V contains at least one exponential by Lemma 9, and thus there is a unique exponential m contained by V. Since $\dim W < \dim V$, it follows from the induction hypothesis that every element of W is of the form $p \cdot m$, where p is an admissible polynomial. Let r_i and s_i be admissible polynomials such that $b_i = r_i \cdot m$ and $p_i = s_i \cdot m$ for every $i = 1, \dots, k$. Put $P = f_0/m$, then $P \in \mathcal{A}$ by (1). Dividing (6) by $m(x+y) = m(x)m(y)$ we obtain

$$P(x+y) - P(x) - P(y) = \sum_{i=1}^{k} r_i(x) \cdot s_i(y) \tag{7}$$

for every $x, y \in G$. Thus $\Delta_y P = P(y) + \sum_{i=1}^{k} s_i(y) \cdot r_i$ for every y. Since $\sum_{i=1}^{k} s_i(y) \cdot r_i$ is a polynomial of degree at most $N = \max_{1 \le i \le k} \deg r_i$, we have $\Delta_{y_1} \dots \Delta_{y_N} \Delta_y P = 0$ for every $y_1, \dots, y_k, y \in G$. We obtain that P is a generalized polynomial. Applying (7) with $y = x$ we find that $P(2x) - 2P(x) = \sum_{i=1}^{k} r_i(x) \cdot s_i(x)$ is an admissible polynomial. Then, since $P \in \mathcal{A}$, Lemma 8 gives that P is an admissible polynomial. (Note that $\mathcal{A}$ contains the constant functions as $m \cdot m^{-1} = 1 \in \mathcal{A}$.) Since $f_0 = P \cdot m$ and $m \in V$, it follows that $f_0 \in W$, contrary to our hypothesis. This final contradiction gives $W = V$, completing the proof. $\square$

Remark 10. We show that Theorems 2, 3 and (ii) of Theorem 4 are easy consequences of Theorem 1.

We prove the 'if' part of Theorem 2, the 'only if' part being simple. Suppose that f is a discrete polynomial, and all translates of f belong to $\mathcal{A}$. Let V_f denote the linear span of the translates of f, and let $\mathcal{A}_f$ denote the algebra generated by V_f. Then $\mathcal{A}_f \subset \mathcal{A}$, and every element of $\mathcal{A}_f$ is a discrete polynomial. Therefore, the only exponential contained in $\mathcal{A}_f$ is the identically 1 function. Consequently, condition (1) is satisfied by $\mathcal{A}_f$.

Now V_f is of finite dimension, since f is a discrete polynomial. Since $V_f \subset \mathcal{A}_f$, it follows from Theorem 1 that f is an admissible polynomial with respect to $\mathcal{A}_f$ and then also with respect to $\mathcal{A}$.

Next we consider Theorem 3. We prove that if $\mathcal{A} \in \mathcal{P}_G$, $\mathcal{A} \neq 0$ and $\mathcal{A}$ is translation invariant, then $\mathcal{A}$ is generated by a set of additive functions and the constant functions. This follows from Theorem 2. First, $\mathcal{A}$ contains the constants, since already V_f contains the constants for every $f \in \mathcal{A}$, $f \neq 0$. If $f \in \mathcal{A}$, then the translates of f belong to $\mathcal{A}$, and thus f is admissible. It is clear that every admissible function is in the algebra generated by the set of additive functions belonging to $\mathcal{A}$ and the constant functions.

As for (ii) of Theorem 4, let $G = \mathbb{C}^n$, and let $f \in \mathbb{C}[x_1, \dots, x_n]$. Consider f as a function from $\mathbb{C}^n$ to $\mathbb{C}$. Then f is a polynomial on G, since $x_1, \dots, x_n$, as functions defined on G, are additive. If $\mathcal{A}_f$ denotes the algebra generated by the translates of f then, by Theorem 2, f is an admissible polynomial w.r.t. $\mathcal{A}_f$. That is, $f = Q(\alpha_1, \dots, \alpha_k)$, where $Q \in \mathbb{C}[x_1, \dots, x_k]$ and $\alpha_1, \dots, \alpha_k$ are additive functions belonging to $\mathcal{A}_f$.

Since $\mathcal{A}_f \subset \mathbb{C}[x_1, \dots, x_n]$, it follows that $\alpha_1, \dots, \alpha_k \in \mathbb{C}[x_1, \dots, x_n]$. In particular, $\alpha_1, \dots, \alpha_s$ are continuous on $\mathbb{C}^n$. Now, if a function $\mathbb{C}^n \to \mathbb{C}$ is additive and continuous then, necessarily, it is a linear function. Therefore, $\alpha_1, \dots, \alpha_k \in \mathcal{A}_f \cap \Lambda_n$, which proves (ii) of Theorem 4.

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